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Traditional instruction of differential equations and conceptual learning

Selahattin Arslan*
Karadeniz Technical University, Fatih Faculty of Education, Primary School Mathematics Teacher Training, Trabzon, Turkey

Email: selaharslan@yahoo.fr

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Procedural and conceptual learning are two types of learning, related to two types of knowledge, which are often referred to in mathematics education. Procedural learning involves only memorizing operations with no understanding of underlying meanings. Conceptual learning involves understanding and interpreting concepts and the relations between concepts. The relationship between these learning types has been discussed for a long time. For some researchers, procedural knowledge forms the basis for conceptual knowledge, while for others the relationship is reversed. The aims of the study reported here were first to explore the nature of students' learning in traditional differential equations (DEs) courses and second to clarify the relationship between procedural and conceptual learning. To address these aims an achievement test with 13 open-ended questions, probing procedural and conceptual learning with regard to DEs, was administered to 77 candidate mathematics teachers, enrolled in a traditional DEs course. The analysis of students' responses to the test items showed that 85% of candidate teachers gave correct responses on procedural questions whilst only 30% of them gave correct responses to the conceptual questions. These findings suggest that the candidate teachers' learning was primarily procedural in the context of traditional instruction and content and that this did not lead them to develop the conceptual knowledge needed to interpret new situations properly and to produce new ideas beyond the ones they had memorized. In addition, based upon the student levels in both procedural and conceptual learning, it was concluded that conceptual learning supports and generates procedural learning but procedural learning does not support conceptual learning.

1. Introduction

Since differential equations (hereafter called DEs) are used to model and understand real life problems; they have an important role in a variety of domains such as Economics, Physics, Biology, etc. This topic is also important for mathematics. In fact, it has strong interrelations with many mathematical concepts including functions, derivatives, integrals, etc. Therefore, students should understand these concepts in order to understand DEs and vice versa; if they understand DEs conceptually, they will understand these concepts better.

Due to the importance of DEs, the topic has attracted many researchers, and the literature shows that a number of researches (e.g. Artigue, 1989; Blanchard, 1994; Boyce, 1994; Chau & Pluvinage, 1999; Kallaher, 1999; Habre, 2000; Rasmussen, 2001; Kwon *et al.*, 2003; Sağlam, 2004; Arslan, 2005; Raychaudhuri, 2008) and projects (Consortium for Ordinary Differential Equations Experiments in Harvey Mudd College; The Boston University Ordinary Differential Equations Project) have been carried out on the teaching of DEs. Nevertheless, most of these studies are about new approaches to teaching DEs and several researchers have pointed out a shortage of studies on students' understanding of DEs (Kwon, 2002; Habre, 2003; Rowland & Jovanoski, 2004; Raychaudhuri, 2008).

2. Procedural learning versus conceptual learning

According to Hiebert and Lefevre (1986), mathematical knowledge is divided into procedural knowledge and conceptual knowledge. They also state that in general terms procedural knowledge is to know *how* and conceptual knowledge is to know *why* something happens in a particular way. This distinction between procedural and conceptual knowledge in mathematics leads to learning being classified as procedural learning or conceptual learning (see Baroody, 2003).

Procedural knowledge is composed of definitions, theorems, rules and algorithms, which render mathematical operations possible and which are presented in everyday language and in other representations (Hiebert & Lefevre, 1986). The New York State Education Department (NYSED) (2005) defines procedural knowledge (procedural fluency) as the skill of handling operations flexibly, fluently, effectively and appropriately. Therefore 'teaching for procedural knowledge means teaching definitions, symbols, and isolated skills in an expository manner without first focusing on building deep, connected meaning to support those concepts' (Engelbrecht *et al.*, 2005, p. 703).

On the other hand, conceptual learning occurs as a result of a 'combination' of existing knowledge and newly encountered knowledge and it enables individuals to understand and appropriate new knowledge. According to NYSED (2005), conceptual knowledge is composed of inner relations formed by associating existing knowledge and includes the individual's understanding of mathematical concepts and relationships and also the individual's knowledge of basic calculation. Therefore, a student who has learned conceptually recognizes and applies definitions, principles, rules and theorems and can compare and contrast related concepts (Engelbrecht *et al.*, 2005).

Conceptual and procedural learning and the relationships between them have been discussed by a number of researchers. According to Piaget, conceptual and procedural knowledge are integral and complementary and the latter forms a basis for the former (cited by Baker & Czarnocha, 2002). On the other hand, Baker and Czarnocha (2002) state that students' conceptual learning improves independently from procedural learning; Hiebert and Lefevre (1986) state that procedural knowledge is meaningful only when it is associated with a conceptual basis. Finally, according to Vygotsky (1986), learning occurs not with repeated calculations (i.e. procedural knowledge) but as a result of reflections upon unconscious conceptual knowledge.

As it can be understood from the definitions above, procedural and conceptual learning are interrelated concepts and both are essential for mathematics. Brown *et al.* (2002) argue that both learning types are important and an individual who has conceptual learning must also have procedural learning but not vice versa. Moreover, many studies conducted in different fields of mathematics state that students who have learned conceptually are more successful than students who have learned procedurally (e.g. Pesek & Kirshner, 2000; Chappell & Killpatrick, 2003).

The literature cited above suggests that conceptual learning and effective learning are equivalent and that conceptual learning will bring about procedural learning. In fact, Engelbrecht *et al.* (2005) claim that the main aim of the reform in analysis instruction is improving conceptual learning. In addition, backed by Haapasalo and Kadijevich (2000), they claim that the reform's principal philosophy is 'students' reaching the state of conceptual learning will bring about procedural learning.' (p. 703).

These observations raise the question of the extent to which conceptual learning occurs through current curricula. Many studies have determined that procedural learning is favoured in analysis courses. Baker *et al*. (2004), for example, state that procedural learning was favoured in traditional curricula because it is believed that learning occurs as a result of operations stemming from suitable procedures. This claim was also supported by Engelbrecht *et al.* (2005). In addition, Aspinwell and Miller (1997) discussed that students completed traditional calculus courses with a little conceptual learning since they were satisfied with procedural learning.

This suggests that it would be interesting to investigate the situation in the case of DEs instruction. In other words, to address the question: Which learning type is favoured in traditional DEs courses?

3. Purpose of the study

A review of the literature shows that a number of researchers and projects have been conducted on the teaching of DEs. Nevertheless, there have been no studies investigating the effects of the traditional instruction focusing on algebraic solution of DEs on conceptual and procedural learning of students.

The aims of the present study are twofold. First, to explore which learning type, conceptual or procedural, predominates in students taking traditional DEs courses, and second to make a contribution to the discussion of the relationship and interaction between procedural and conceptual learning.

4. Methods

4.1. Sample

The study was carried out with 77 candidate teachers studying at the Mathematics Education Department at Fatih Faculty of Education in Karadeniz Technical University (Turkey) and taking DEs courses. The content of the course was determined through an analysis of the lecture notes of the instructor and is outlined as follows.

- What is a DE.
- What do a particular solution and the general solution mean.
- How to obtain a particular solution from the general solution.
- How to solve various types of first order DE (linear, homogenous, exact, etc.) and how to convert a DE into one of these types.
- The Existence and Uniqueness Theorem.
- Riccati, Bernouilli, Clairaut, etc. DEs and how to solve them.
- First order non-linear DEs and their solutions.
- Applications of first and second order DEs.
- How to obtain the DE of a family of curves.
- Orthogonal curve orbit of an orbit and how to obtain it.
- That y' stands for the slope of the tangent line and how to express concepts like length of sub-tangent, of subnormal, etc. of a curve by a DE.

Traditional DEs courses are dominated by algebraic solutions. In other words, generally DEs that can be easily solved algebraically, and carefully chosen applications of these equations are preferred in traditional courses. In some contemporary DEs courses, the stress on algebraic approaches is reduced and more numerical and graphical approaches are included with the intent of facilitating conceptual learning of DEs by students. Given the content outlined above of the course taken by the students participating in this study, it is reasonable to describe it as a traditional DEs course.

TABLE 1. *Comparison between questions measuring conceptual and procedural knowledge*

Some of the characteristics that a question measuring procedural knowledge has:	Some of the characteristics that a question measuring conceptual knowledge has:
(i) it consists of an application of existing knowledge	(i) it requires understanding newly encountered ideas using existing knowledge
(ii) the knowledge required by the question was taught previously	(ii) the knowledge required by the question was not previously taught
(iii) the task given in the question has already been discussed in the class	(iii) the task given in the question has not been discussed in the class
(iv) it involves making calculations step by step just as was taught	(iv) it provides students with the opportunity to respond flexibly
(v) it requires cognitive skills at the knowledge level of Bloom's taxonomy (theorems, definitions, etc.)	(v) it requires cognitive skills at the higher levels of Bloom's taxonomy
(vi) it involves only mechanical application of the knowledge taught. While answering such questions, the individual generally makes use of memorized knowledge without needing to understand the concept thoroughly and without applying existing knowledge in different learning environments	(vi) it involves perceiving the concepts deeply and associating different concepts with each other. While answering such questions, students form associations with their existing knowledge, express their knowledge using different representations (oral, graphical, etc.), demonstrate the ability to shift between these representations, and apply their existing knowledge in different situations even if it is in another domain of mathematics

4.2. Instruments

An achievement test with 13 open-ended questions measuring both procedural and conceptual learning was used as a data collecting tool in this study. As it is not always easy to classify pre-existing questions according to which learning type they address, the characteristics of each learning type were determined based upon the literature review and then the questions in the test were prepared.

In order to determine characteristics of each learning type, we made use of previous research and especially the works of Engelbrecht *et al*. (2005), National Council of Teachers of Mathematics (2000), Skemp (1987) and the characterization scale developed by Baki and Kartal (2004). On this basis Table 1 was prepared, which contrasts procedural and conceptual learning and which also makes the similarities and differences more obvious.

On the basis of these criteria, five questions (labelled as C1, C2, C3, C3bis, and C4) in the achievement test were designed to measure conceptual learning, while eight questions (labelled as P1, . . . , P8) were designed to measure procedural learning.

The procedural and conceptual questions were given to students in a mixed order, but they are presented below separately for the sake of clarity. These questions measure procedural knowledge:

P1: Is the function $xy^2 + \frac{x^3}{3} = 1$ a particular solution to the DE $(x^2 + y^2)dx + 2xydy = 0$?
P2: Write down the DE of the family of curves whose slope of the tangent line at any point is equal to the ordinate of that point.
P3: What is a separable DE? Give an example. How do you solve?
P4: In each case, indicate how you solve the given DE:
P4a: Homogenous, P4b: Bernouilli, P4c: Riccati, P4d: Linear
P5: Solve the DE $y' = ((xy^2 - 1)/(1 - x^2y))$.
P6: Solve the DE $y' = e^{2x - y}$.
P7: Solve the DE $(x^3 + y^3)dx + 3xy^2dy = 0$.
P8: What does the DE $y' = x$ mean geometrically?

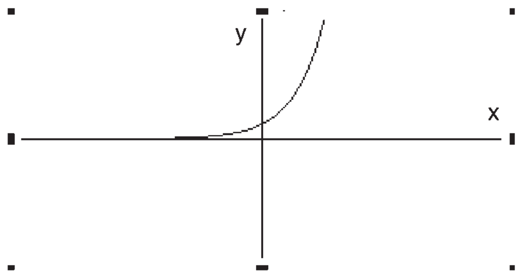

FIGURE 1. Curve given for questions C3 and C3bis.

Five of the questions above (P3, P4, P5, P6 and P7) were related to 'solutions of first order ordinary DEs', which is one of the indispensable tasks of the traditional DEs courses. In this context, the students were asked to describe what separable, homogenous, Bernouilli, Riccati, and Linear DEs are and how they could be solved. In addition, students were given and expected to solve DEs which were associated with at least one of these types (exact and homogenous in P5 and P7; and exact and separable in P6).

Questions P2 and P8 are both about the geometric interpretation of y'.

When the questions in this group and the course content given above are compared, it can be said that these questions fit with what was taught to the students in the course and they fulfil the criteria for procedural learning. For this reason, these questions are interpreted to measure only procedural knowledge as they do not require students to understand DEs deeply or to go beyond existing knowledge to answer them.

The questions measuring conceptual knowledge are presented as follows:

C1: Could the curve above (Figure 1) be a solution to the DE $y' = 2y$?
C2: Could a given function $f(x,y)$ be a particular solution of different DEs? Justify your answer.
C3: Could a given curve be a solution of different DEs? Justify your answer.
C3bis: Could the curve given above (Figure 1) be a solution to the DE $y' = y^2$?
C4: Find the equation of the tangent line at $(1, 1)$ of the solution curve of the DE $y' = y$, passing through $(1, 1)$.

These questions focus on whether a function or a curve can be a solution of two different DEs, obtaining the equation of the tangent line at any arbitrary point of a solution curve and similar issues.

Question C1 is similar to P1, but unlike P1, a curve is given with no algebraic expression making an algebraic solution impossible. This makes C1 suitable for a student with a conceptual understanding. In this context, $y' = 2y$ could be interpreted qualitatively and the regions where solution curves increase and decrease could be found. Or students might calculate the slopes for the tangent lines at different points of the curve with the help of the DE and then compare this value with the graph.

Questions C2 and C3 are similar. Such questions were not discussed in the course; however, students could reach solutions in various ways by conceptually interpreting the knowledge they had. They knew of many functions that are particular solutions of different DEs. The function $f(x) = 0$, for example, is a particular solution to DEs like $y' = y$; $y' = 2y$ etc. And e^{2x} is a particular solution of two DEs with the general solutions $Ae^{2x} + Be^{-3x}$ and Ce^{2x} (A, B, C$\in\Re$). In this way, indefinite number of common solutions can be found by letting B = 0 and A = C. Similarly, e^x, is a particular solution for DEs like $y' = y$ and $y'' + y' = 2y$.

Question C3bis was proposed as the continuation of C3. In that, a sample case is given. This question was asked both to facilitate students' understanding of C3 and to reveal students' reasoning.

Question C4 is closely related to questions P2 and P8 and concerns finding the equation of the tangent line by means of the DE given. Since the slope of the tangent at (1, 1) of the curve mentioned can be calculated as 1 from the DE, the question becomes finding the equation of a line through the point *(1, 1)* with the slope 1, which the students will have learned to do in school. The important point here is applying information given in the course, namely: '*Geometrically* $y' = y$ *gives the slopes of the tangents lines at any point*.' to reach the conclusion '*The slope of the tangent line of the solution curve of* $y' = y$ *passing through the point (1, 1) is* $y' = y \Rightarrow y' = 1$.' This question could also be solved by finding the solution curve of the DE $y' = y$ passing through the point *(1, 1)*.

As can be seen, the questions in this group require students to interpret existing knowledge and be flexible, to relate different concepts and to use logic flexibly. For these reasons, among others, these questions can be said to measure conceptual learning.

4.3. *Data analysis*

The data obtained were analyzed using categories and then level determination methods.

The students' answers were categorized as *correct*, *wrong* and *no answer*, by considering the procedures the students used while they were answering the questions. During this categorizing process, the answers of the students who used the correct procedure but could not complete the solution or made a sign mistake were accepted as correct. Based on these categories, student achievement on the procedural and conceptual questions was compared. The answers to related questions of each type (e.g. C1 and P1) were also compared to have a closer look at the relationship between procedural and conceptual learning.

In addition, the questions were graded on a scale designed so that a student could reach a maximum score of 100 for each of the learning types (procedural and conceptual) and two different scores were calculated. Each 20 point range was taken as a level resulting in five levels of achievement, from level 0 to 4. For example, a student with a score of 69 on the procedural questions and 37 on the conceptual questions is scored at level 3 for procedural and level 1 for conceptual learning. Each student's levels were determined and these levels were compared to investigate the relationship between procedural and conceptual learning.

5. Results and discussion

5.1. *Questions related to procedural learning*

The data for the answers to the eight questions in the procedural group are summarized in Table 2. The table shows that a very high percentage (average 85%) of the candidate teachers answered these questions correctly. On an average, 10% of the candidate teachers gave wrong answers for these questions and 5% of them did not provide any answer.

These questions will be analyzed separately with respect to achievement in descending order.

5.1.1. P7: Resolution of the exact and homogenous DE

Table 2 shows that the largest number of students were successful in answering P7 (96%) which involves a DE that can be solved in two different ways as it is exact and homogenous. Only two students answered this question incorrectly; one did not answer.

TABLE 2. *Achievement of students for the questions in the procedural category*

Questions→ Status↓	P1	P2	P3	P4	P5	P6	P7	P8	Percentage
Right frequency (%)	65 (84)	62 (81)	72 (94)	65 (85)	59 (77)	63 (82)	74 (96)	59 (77)	85
Wrong frequency (%)	6 (8)	11 (14)	2 (3)	8 (10)	14 (18)	7 (9)	2 (3)	12 (16)	10
No answer frequency (%)	6 (8)	4 (5)	3 (3)	4 (5)	3 (3)	7 (9)	1 (1)	4 (5)	5

5.1.2. P3 and P6: Resolution of separable DEs

Only slightly fewer students (94%) were successful in answering P3 which asks for a definition of a separable DE and how to solve one. But fewer (82%) answered P6 correctly, even though it is a similar question. P6 asks for the solution of $y' = e^{2x-y}$ which is a separable DE. When the students' answers were examined, it was determined that this difference in student success arose from students' inability to identify the equation type. A detailed analysis of the student answers showed that some of the students tried to convert this equation into exact form, some tried to solve it with $y = vx$ as is done for homogenous equations and still others tried to make the equation more familiar by applying the conversion $2x - y = u$.

5.1.3. P5: Resolution of the exact DE

Only 77% of the students correctly answered P5, which asks for the solution of the exact DE: $y' = ((xy^2 - 1)/(1 - x^2y))$. Analysis of the responses indicated that again the students could not fully identify the equation type. As a matter of fact, some of the students who gave wrong answers tried to treat the equation like a linear one, others confused it with a Bernouilli equation and tried to solve it with the conversion $v = y^{-2+1}$ or confused it with a homogenous equation and tried to solve it with the conversion $xy = z$.

5.1.4. P4: Resolution of the homogenous, Bernouilli, Riccati and linear DE

In the four parts of P4, asking what homogenous (P4a), Bernouilli (P4b), Riccati (P4c) and Linear (P4d) DEs are and how to solve them, the students answered correctly with the 88, 88, 80 and 79%, respectively, with an average of 85% for P4 overall. For P4a, 6% gave an incorrect answer. These students thought that the conversion $y = vx$ converts the equation into a linear equation instead of a separable equation or they confused it with an exact or linear equation solution. For P4b, 11% gave a wrong answer. Some students confused the equation with a linear equation and others stated that the conversion $v = y^{-n+1}$ converts it into a separable equation instead of a linear equation. Among the 12% of students who gave a wrong answer to P4c, there were students who claimed that it can be converted into a linear or separable equation and there were students who confused it with a Bernouilli equation. For the linear equation question P4d, 15% gave an incorrect response, including some students who confused it with equations that could be converted into exact ones and other students who stated that the equation could be separable when it was multiplied by an integrating factor.

5.1.5. P1: Particular solution of a DE

On question P1, which asked whether $xy^2 + \frac{x^3}{3} = 1$ was a particular solution of the DE $(x^2 + y^2)dx + 2xydy = 0$, 84% of the students were successful. They applied a number of different solution strategies, including obtaining the DE given by deriving the function and proving that the given function could be reached by giving a special value to the parameter C in the general solution of the DE.

Incorrect answers were given by 8% of the students. Some made mistakes while deriving and others while solving the equation.

5.1.6. P2 and P8: Geometrical interpretation of a DE

In one of the questions associated with the geometrical interpretation of *y'* in a given DE (P2), 81% of the students gave the right answer $y' = y$ and 5% did not answer the question at all. The students who gave wrong answers (14%) provided 6 different answers. For the most part, these students gave meaningless answers. For example, some of the students took $y = \frac{dy}{dx}x$ and found '*slope* $= \frac{dy}{dx} = \frac{y}{x}$' and then considering that the slope is equal to the ordinate wrote $\frac{dy}{dx} = \frac{y}{x} = y$ and arrived at $dy = ydx$.

In the case of P8, which is conceptually related to P2, there was a slight decrease in the success rate from 81% to 77%. Among the wrong answers given, there were meaningless answers like '*The equation is a parabola with the slope of y' = x*' and answers indicating that students had difficulty in understanding DEs. At the same time, there were some answers indicating that students have difficulties in stating the relation between the slope of the tangent line and the length of the sub-normal. For example:

'[Since] $y' = x$ [then] $\frac{dy}{dx} = x$. [So,] *the equation* $x\ \frac{dy}{dx} = 1$ *can be obtained. And this equation stands for the length of the sub-normal. So, the sub-normal for this curve is 1.*'

Analysis of the answers given to the procedural learning questions suggested that students were very successful in answering these questions but at the same time the explanations the students gave for their answers indicate that the students have difficulty in perceiving DEs conceptually.

5.2. Questions related to conceptual learning

Table 3 shows that the student success rate for the questions measuring conceptual learning was 30% on an average. This rate was much lower than the rate for procedural learning (85%). Of the students, 41% gave wrong answers and 29% of the students preferred not to give any answers for this group of questions.

5.2.1. C1: Qualitative interpretation of a DE

As is shown in Table 3, 70% of the students gave correct answers for C1. When compared with the rest of the questions measuring conceptual learning, the success rate for this question was quite high. When the students' answers were analyzed, it was found that the students applied three different strategies to this question. Some students (24) associated an algebraic equation (e^{2x}) to the given graph and proved that this function satisfies the DE $y' = 2y$ (strategy S1). By applying this strategy, students actually altered the question into one with which they were acquainted (see P1). A second strategy was used by

TABLE 3. *Achievement of the students for the questions in the conceptual category*

Questions→ Status↓	C1	C2	C3	C3Bis	C4	Percentage
Correct (f-%)	54 (70)	17 (22)	6 (8)	20 (26)	18 (23)	30
Wrong (f-%)	11 (14)	55 (71)	13 (17)	22 (29)	56 (73)	41
No answer (f-%)	12 (16)	5 (7)	58 (75)	35 (45)	3 (4)	29

27 students; solving $y' = 2y$, to obtain the general solution ($y = Ce^{2x}$) and stating that the curve could be obtained for a special value of C (C = 1 for instance) (strategy S2). Finally, three students decided, by checking the slope of the tangent at a single point (for example $y = 0$) or by using vague expressions like '*it is possible if the slope at any point is 2y*', that the graph could show a solution to the given DE (strategy S3). Here it should be stated that S1 and S3 are not mathematically sufficient since they are not precise enough. However, they still guided the students to the correct answer and that is why they were counted as correct. This makes the success rate for this question higher. If only the mathematically complete strategy, S2, is accepted as right, the success rate is reduced to 35%.

5.2.2. C2: Whether a function can be a solution of different DEs

Only 22% of the students gave a correct answer for C2, while 71% of them gave wrong answers. Some of the students who provided wrong answers noted that a single function could not be a solution for different DEs without giving any supporting reason and some others said that it was possible but put forward invalid explanations like: '*a DE has indefinite numbers of solutions. So, solution sets of two different DEs might have common member*.' Such explanations imply that students did not understand conceptually the C parameter in the general solution. It is true that the general solution of $y' = y$ is $y = Ce^x$ and that this has an infinite number of solutions. Similarly, the general solution of $y' = \frac{1}{x^2}$ is $y = C - \frac{1}{x}$ and this equation also has an infinite number of solutions. But these two infinite sets do not intersect and the two DEs do not have even a single common solution. This suggests that the students confuse the fact that 'the general solution has indefinite number of solutions' with 'the general solution includes all types of functions.'

5.2.3. C3: Whether a curve can be the graph of the solution of different DEs

Question C3, which is similar to C2, asked the students to discuss whether any curve could be the graph of the solution of more than one DE. One student put forward an acceptable qualitative interpretation. Another student stated '*I reached the same result by solving different DEs before*' and 4 students were satisfied with only saying 'perhaps' and did not provide any explanation. These six answers were counted as correct. Many students (75%) did not attempt this question and 17% of the students gave a wrong answer. The wrong answers clearly showed that conceptual learning had not occurred. Some examples of wrong answers are as follows:

'*It is not possible because two DEs with the same curves would be identical*',
'*Because the slope of the curve and points on it are specific to the DE*',
'*If their slope and their crossing points are the same, the equations are identical*'.

5.2.4. C3bis: Relationship between a curve and a DE

The success rate on this question, which asked whether the curve given in C1 (i.e. Figure 1) could also show a solution of $y' = y^2$ was 26%, with 45% not attempting the question. This question was answered with 20 different types of wrong answers which indicated that conceptual learning had not occurred. For example, although some of the students found the general solution of the equation $y' = y^2$, they made explanations like these:

'*Here y can take negative values. However they are not in the graph*' or
'*In the solution curve of this equation* [general solution?] *$y < 0$ is possible but it does not happen in the graph. So, it is not possible*.'

Such explanations showed that students had a misconception like: '*Just as two different functions should have different curves, a DE has a unique curve and this later is specific to the equation.*' A similar misconception was also found in a study conducted in France (Arslan, 2005).

The high rate of not answering for C3 and C3bis might arise from that fact that the questions are complementary. The students might think that answering only one of them would be enough.

5.2.5. C4: Finding the slope by means of a DE

The success rate for C4 was 23%. Of the students, 73% gave wrong answers to this question and 4% of them left the question unanswered. The analysis of the answers indicated that the main reason students' had the inability to answer this question was that they did not know how to find the slope by using a DE. Most of the students wrote the equation of the tangent line as $y - 1 = m(x - 1)$ but had difficulty in determining the slope (m). Some of the students assumed $m = y$ based on $y' = y$, while some others went one step ahead and put Ce^x instead of m. Still other students took y' instead of m and obtained another DE which they then solved.

As C2 is related to C3 and C1 to C3bis, a comparison between them may be interesting. While the success rate in C2 was 22%, it was about 8% for C3. The reason for this difference might be the inclusion of very few graphical representations in traditional DEs courses as question C3 is based on a graph. The difference in the success rates for C1 and C3bis is even greater (70% versus 26%). The main reason for this difference might be that the strategies students used for the solution of C1 did not yield solutions for C3bis. For example, in strategy S1, the equation of the curve is taken to be e^{2x} which satisfies the DE in C1; however, it does not satisfy the equation given in C3bis. The other two strategies do not work for C3bis, either. This proves once again that strategies S1 and S3 used for solving C1 were not precise.

6. Comparison: procedural and conceptual learning

In broad terms, it can be seen that students were more successful on questions measuring procedural learning than questions measuring conceptual learning. It is interesting to compare the related questions in the conceptual and procedural categories. Recall that questions P2 and P8 are related to question C4, and P1 is related to C1. From Tables 2 and 3, it can be seen that the success rates for P2, measuring the ability to express a geometrical feature with a DE, and P8, asking for a geometrical interpretation of an ordinary DE, were 81 and 77%, respectively. However, for the related question C4, the success rate is only 23%. From this, we can conclude that the students know a geometrical interpretation of the equation $y' = y$ but cannot use it. In the other case, the success rate for question P1 was 84% versus 70% for question C1, and if only strategy S2 is counted as correct, it is reduced to 35%.

7. The relationship between procedural and conceptual learning

In order to investigate the relationship between these two types of learning, the levels determined by the students' scores were examined. They are summarized in Table 4. Table 4, panel a shows the conceptual level of students with respect to their procedural level and panel b shows their procedural level with respect to their conceptual level. For example, Table 4, panel a shows that there are 8 students at level 2 according to the procedural questions and that 3 of them are at level 0 according to the conceptual questions while 3 are at level 1, one student is at level 2 and one of them is at level 3.

TABLE 4. *Comparison between students' levels: procedural versus conceptual (panel a) and conceptual versus procedural (panel b)*

Procedural [Level]/frequency	Conceptual [Level]/frequency
Panel a: Procedural → Conceptual	
[0]/3	[0] 1
	[1] 2
[1]/1	[3] 1
[2]/8	[0] 3
	[1] 3
	[2] 1
	[3] 1
[3]/22	[0] 7
	[1] 5
	[2] 7
	[3] 3
[4]/43	[0] 9
	[1] 14
	[2] 19
	[3] 1
Conceptual [Level]/frequency	**Procedural [Level]/frequency**
Panel b: Conceptual → Procedural	
[0]/20	[0] 3
	[1] 1
	[2] 8
	[3] 8
[1]/24	[3] 14
	[4] 10
[2]/27	[4] 27
[3]/6	[4] 6
[4]/0	

From Table 4, panel a, it is evident that the procedural level of the students does not predict their conceptual level. As a matter of fact, among the students having a low procedural level, there were students with a high conceptual level as well as students with low conceptual level. Similarly, there were students with both low and high conceptual levels among the students with a high procedural level. For example, 23 of the 43 students with the highest procedural level were at conceptual levels 0 or 1 and 20 of them were at levels 2 or 3. Thus, the data presented in Table 4, panel a show that it cannot be said that procedural learning predicts conceptual learning. This result does not support the thesis that claims that procedural learning provides a basis for conceptual learning or that an individual who has learned procedurally will also have learned conceptually.

On the other hand, Table 4, panel b shows that the level of success on the conceptual questions does predict success on the procedural questions. According to the table, students with low conceptual level also had low procedural level and students with high conceptual level also had high procedural level. The highest procedural level reached by a student on conceptual level 0 was procedural level 3 and the students on conceptual level 3 or even 2 were all at the top procedural level. This shows that as the conceptual level gets higher, the procedural level also gets higher. This supports the idea that an individual who has learned conceptually will also be able to answer procedural questions.

The distribution of the students in the levels shows that the students were more successful in the procedural tasks, which supports the data from the analysis presented above. Referring to Table 4, panel a, there are students on each of the five levels but they are mostly on higher levels. In fact, 65 of the students are in the top two levels and got a score of 60 or higher on the procedural questions. Only four of the students were in the bottom two levels and got scores below 40. However, as is shown in Table 4, panel b, there were no students getting scores above 80 on the conceptual questions and 44 students were in the bottom two levels and got scores below 40 on the conceptual questions.

8. Conclusion

This study aimed to determine whether procedural or conceptual learning is predominant in a traditional DEs course and to investigate the relationship between these two types of learning. To address these aims, eight questions measuring procedural learning and five questions measuring conceptual learning were given to 77 candidate teachers taking a traditional DEs course.

While the success rate for the questions measuring procedural learning was about 85%, it was only 30% for the questions measuring conceptual learning. In addition, when we examine the grades that the students had, 65 of the students attained over 60 for procedural questions while none of the students attained above 80 on the conceptual questions and only six attained above 60. On the basis of this result and the incorrect and meaningless expressions the students used in their answers to conceptual questions, it can be said that little conceptual learning occurred and the students have difficulty in perceiving DEs conceptually. Therefore, it can be said that students' learning was procedural in their traditional DEs course and was limited to mastering and applying some algebraic techniques. These results support the results of the many previous studies that found that students have misconceptions and learning difficulties about DEs (Artigue, 1989; Boyce, 1994; Rasmussen, 2001). Saglam (2004) and Anderson & Seaquist (1999), for example, remarked that students have difficulties in understanding what a DE expresses and so in interpreting a DE modelling a problem from everyday life.

At the same time, this study found that level of procedural learning did not imply level of conceptual learning but that level of conceptual learning did imply level of procedural learning. Therefore, it is not possible to predict conceptual success of students based on their procedural success but it is possible to predict procedural success on the basis of their conceptual success. This supports the idea that procedural learning does not guarantee conceptual learning, but rather conceptual learning supports procedural learning (Brown *et al.*, 2002).

It is known that there are three different approaches to solving DEs: Algebraic (analytic), numerical and qualitative (graphical) (Artigue, 1989; Arslan, 2008). While the algebraic approach predominates in traditional DEs courses, in contemporary DEs courses the emphasis on algebraic approaches is reduced and numerical and graphical approaches are included more and in this way the conceptual learning of DEs by students is facilitated. For these reasons, many researchers have highlighted different approaches to the instruction of DEs and the necessity of these approaches (Artigue, 1992; Boyce, 1994; Arslan, 2005). The results of this study, investigating the effects of a traditional DEs course in terms of conceptual and procedural learning, imply that traditional instruction is not sufficient particularly for conceptual learning and reveals once again the need for contemporary approaches.

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Selahattin Arslan is an assistant professor in Primary Mathematics Education Department in Karadeniz Technical University, Trabzon, Turkey. He received his PhD degree from the University of Joseph Fourier in Grenoble, France, in 2005. He is interested in ICT, Teacher Education and Mathematics Education, particularly differential equations.